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Supersymmetric renormalization prescription in $\mathcal{N}=4$ super-Yang–Mills theory

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Abstract

Using the shadow dependent decoupled Slavnov–Taylor identities associated to gauge invariance and supersymmetry, we discuss the renormalization of the $\mathcal{N}=4$ super-Yang–Mills theory and of its coupling to gauge-invariant operators. We specify the method for the determination of non-supersymmetric counterterms that are needed to maintain supersymmetry.

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1 Introduction

Non-linear aspects of supersymmetry and the non-existence of a supersymmetry-preserving regulator make the renormalization of supersymmetric theories a subtle task. Whichever is the choice of regularization, we expect non-supersymmetric counterterms for maintaining supersymmetry at the renormalized level. A very effective regularization of UV divergences of super-Yang–Mills theories, called dimensional reduction, was introduced quite early by Siegel [1]. Whether this regularization holds true at all orders in perturbation theory was questioned in [2]. With suitable improvements, its compatibility with the quantum action principle was shown in [3]. In fact, this regularization cannot preserve supersymmetry beyond 3-loop order [4], which implies the introduction of non-supersymmetric counterterms for 4-loop computations. As another complication, the renormalizable Lorentz covariant gauge conditions (Landau–Feynman-type gauges) break supersymmetry. This breaking of a global symmetry is analogous to that of the Lorentz invariance by axial or Coulomb gauges for the ordinary Yang–Mills theories, but it is more intricate, because supersymmetry is realized non-linearly. This question was addressed by Dixon [5, 6], who completed the ordinary BRST symmetry transformations for gauge invariance by adding supersymmetry transformations, whose supersymmetry parameter is a commuting constant spinor. The "enlarged BRST symmetry" determines a Slavnov–Taylor identity. It was shown, in a series of papers by Stöckinger et al., that this process allows the determination, order by order in perturbation theory, of non-invariant counterterms [7] that restore supersymmetry covariance of Green functions in the $\mathcal{N} = 1$ models [8]. The unusual feature that occurs is that Feynman rules depend on the parameter of supersymmetry, but it is advocated that observables do not depend on it. This method has a conceptual backlash. To define the "enlarged BRST symmetry", Dixon changed the transformation law of the Faddeev–Popov ghost (to achieve nilpotency of the "enlarged BRST transformations"). But then, the BRST equation of the Faddeev–Popov ghost loses its geometrical meaning. Moreover, observables are not defined as they should be, from the cohomology of the BRST differential, since, in this case, they would be reduced to supersymmetry scalars. They must be introduced as gauge-invariant functionals of physical fields, which are well defined classically, but are sources of confusion at the quantum level, because of their possible mixing with non-gauge-invariant operators. In fact, the previous methods are sufficient to define certain rules for practical perturbative computations, but the way they are obtained lacks the important feature of relying on a well-funded algebraic construction. The latter must be independent of the renormalization scheme and clearly separates gauge invariance from supersymmetry.

In recent papers, we indicated the possibility of disentangling these two invariances, for defining the quantum theory, with independent Slavnov–Taylor identities [9]. We introduced new fields, which we called shadows, not to confuse them with the usual Faddeev–Popov ghosts. The advantage of doing so is as follows. The obtained pair of differential operators allows us to define the two Slavnov–Taylor operators that characterize the gauge-fixed BRST-invariant supersymmetric quantum field theory, while the Faddeev–Popov ghost keeps the same geometrical interpretation as in the ordinary Yang–Mills theory. Observables are defined by the cohomology of the BRST differential Slavnov–Taylor operator and their supersymmetry covariance is controlled at the quantum level by the other Slavnov–Taylor operator for supersymmetry. This will allow for an unambiguous perturbative renormalization of supersymmetric gauge theories.

The shadow fields are assembled into BRST doublets, and they do not affect the physical sector. The quantum field theory has an internal bigrading, the ordinary ghost number and the new shadow number. The commuting supersymmetry parameter is understood as an ordinary gauge parameter for the quantum field theory. The prize one has to pay for having shadows is that they generate a perturbative theory with more Feynman diagrams. If we consider physical composite operators that mix through renormalization with BRST-exact operators, we have in principle to consider the whole set of fields in order to compute the supersymmetry-restoring non-invariant counterterms. For certain "simple" Green functions, which cannot mix with BRST-exact composite operators, there exist gauges in which some of the additional fields can be integrated out, in a way that justifies the work of Stöckinger et al. in the $\mathcal{N} = 1$ theories. By doing this elimination, we lose the geometrical meaning, but we may gain in computational simplicity.[1]

In the conformal phase of the $\mathcal{N} = 4$ super-Yang–Mills theory, the observables are usually defined as correlation functions of gauge-invariant operators. The aim of this paper is to discuss their quantum definition and the methodology that is needed to non-ambiguously compute non-invariant counterterms and maintain supersymmetry. In fact, our results apply to the renormalization of all supersymmetric theories.

[1]We do not exclude the possibility of also reducing the set of fields in the general case, including observables that mix with BRST-exact operators through renormalization, but further investigations are needed in order to establish this statement.

2 Shadow fields and supersymmetry Slavnov–Taylor identities

2.1 Action and symmetries

The physical fields of the $\mathcal{N}=4$ super-Yang–Mills theory in $3+1$ dimensions are the gauge field A_μ the $SU(4)$-Majorana spinor λ, and the six scalar fields ϕ^i in the vector representation of $SO(6)\sim SU(4)$. They are all in the adjoint representation of a compact gauge group that we will suppose simple. The classical action is uniquely determined by $Spin(3,1)\times SU(4)$, supersymmetry and gauge invariance. It reads

$$S \equiv \int d^4x \mathrm{Tr}\, \Big(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}D_\mu\phi^i D^\mu\phi_i + \frac{i}{2}\big(\overline{\lambda}\not{D}\lambda\big) - \frac{1}{2}\big(\overline{\lambda}[\phi,\lambda]\big) - \frac{1}{4}[\phi^i,\phi^j][\phi_i,\phi_j]\Big) \quad (1)$$

with $\phi \equiv \phi^i\tau_i$ and the supersymmetry transformations δ^{Susy} [2]

$$\delta^{Susy}A_\mu = i\big(\overline{\epsilon}\gamma_\mu\lambda\big) \qquad \delta^{Susy}\phi^i = -\big(\overline{\epsilon}\tau^i\lambda\big) \qquad \delta^{Susy}\lambda = \Big(\not{F} + i\not{D}\phi + \frac{1}{2}[\phi,\phi]\Big)\epsilon \quad (2)$$

Out of δ^{Susy}, we can build an operator Q that also acts on the Faddeev–Popov field Ω and new shadow fields c, μ [9]. Q acts on all the physical fields as $Q=\delta^{Susy}-\delta^{\mathrm{gauge}}(c)$, and we have

$$\begin{aligned} Qc &= (\overline{\epsilon}[\phi - i\not{A}]\epsilon) - c^2 \qquad Q\Omega = -\mu - [c,\Omega] \\ Q\mu &= -[(\overline{\epsilon}\phi\epsilon),\Omega] + i(\overline{\epsilon}\gamma^\mu\epsilon)D_\mu\Omega - [c,\mu] \end{aligned} \quad (3)$$

The BRST operator s is nothing but a gauge transformation of parameter Ω on all physical fields, and we have

$$s\,\Omega = -\Omega^2 \qquad s\,c = \mu \qquad s\,\mu = 0 \quad (4)$$

To define a BRST-exact supersymmetric gauge-fixing, we introduce the trivial quartet $\bar{\mu}$, $\bar{c}$, $\bar{\Omega}$, b, with

$$\begin{aligned} &s\,\bar{\mu} = \bar{c} \qquad s\,\bar{c} = 0 \qquad s\,\bar{\Omega} = b \qquad s\,b = 0 \\ &Q\bar{\mu} = \bar{\Omega} \qquad Q\bar{c} = -b \qquad Q\bar{\Omega} = -i(\overline{\epsilon}\gamma^\mu\epsilon)\partial_\mu\bar{\mu} \qquad Qb = i(\overline{\epsilon}\gamma^\mu\epsilon)\partial_\mu\bar{c} \end{aligned} \quad (5)$$

[2] The parameter ϵ is a commuting spinor, so that ${\delta^{Susy}}^2 \approx \delta^{\mathrm{gauge}}(\overline{\epsilon}[\phi - i\not{A}]\epsilon) - i(\overline{\epsilon}\gamma^\mu\epsilon)\partial_\mu$, where $\approx$ stands for the equality modulo equations of motion.

On all fields, we have $s^2 = 0$, $Q^2 \approx -i(\bar\epsilon\gamma^\mu\epsilon)\partial_\mu$, $\{s, Q\} = 0$. We have the following renormalizable supersymmetric $s\,Q$-exact gauge-fixing actions[3]

$$-s\,Q\int d^4x \mathrm{Tr}\,\Big(\bar\mu\partial^\mu A_\mu + \frac{\alpha}{2}\bar\mu b\Big) \tag{6}$$

By introducing sources associated to the non-linear s, Q and $s\,Q$ transformations of fields, we get the following ϵ-dependent action, which initiates a BRST-invariant supersymmetric perturbation theory[4]

$$\begin{aligned}\Sigma \equiv \frac{1}{g^2}S - \int d^4x \mathrm{Tr}\,\Big(&b\partial^\mu A_\mu + \frac{\alpha}{2}b^2 - \bar c\partial^\mu\big(D_\mu c + i(\bar\epsilon\gamma_\mu\lambda)\big) - \frac{i\alpha}{2}(\bar\epsilon\gamma^\mu\epsilon)\bar c\partial_\mu\bar c\\ &+ \bar\Omega\partial^\mu D_\mu\Omega - \bar\mu\partial^\mu\big(D_\mu\mu + [D_\mu\Omega, c] - i(\bar\epsilon\gamma_\mu[\Omega,\lambda])\big)\Big)\\ + \int d^4x \mathrm{Tr}\,\Big(&A^{(s)}_\mu D^\mu\Omega + \overline\lambda^{(s)}[\Omega,\lambda] - \phi^{(s)}_i[\Omega,\phi^i] + A^{(Q)}_\mu QA^\mu - \overline\lambda^{(Q)}Q\lambda + \phi^{(Q)}_i Q\phi^i\\ &+ A^{(Qs)}_\mu s\,QA^\mu - \overline\lambda^{(Qs)} s\,Q\lambda + \phi^{(Qs)}_i s\,Q\phi^i + \Omega^{(s)}\Omega^2 - \Omega^{(Q)}Q\Omega - \Omega^{(Qs)} s\,Q\Omega\\ &- c^{(Q)}Qc + \mu^{(Q)}Q\mu + \frac{g^2}{2}(\overline\lambda^{(Q)} - [\overline\lambda^{(Qs)},\Omega])M(\lambda^{(Q)} - [\lambda^{(Qs)},\Omega])\Big)\end{aligned} \tag{7}$$

Because of the s and Q invariances, the action is invariant under both Slavnov–Taylor identities defined in [9], which are associated respectively to gauge and supersymmetry invariance, $\mathcal{S}_{(s)}(\Sigma) = \mathcal{S}_{(Q)}(\Sigma) = 0$. The supersymmetry Slavnov–Taylor operator is[5]

$$\begin{aligned}\mathcal{S}_{(Q)}(\mathscr{F}) \equiv \int d^4x \mathrm{Tr}\,\Big(&\frac{\delta^R\mathscr{F}}{\delta A^\mu}\frac{\delta^L\mathscr{F}}{\delta A^{(Q)}_\mu} + \frac{\delta^R\mathscr{F}}{\delta\lambda}\frac{\delta^L\mathscr{F}}{\delta\overline\lambda^{(Q)}} + \frac{\delta^R\mathscr{F}}{\delta\phi^i}\frac{\delta^L\mathscr{F}}{\delta\phi^{(Q)}_i} + \frac{\delta^R\mathscr{F}}{\delta c}\frac{\delta^L\mathscr{F}}{\delta c^{(Q)}} + \frac{\delta^R\mathscr{F}}{\delta\mu}\frac{\delta^L\mathscr{F}}{\delta\mu^{(Q)}}\\ &+ \frac{\delta^R\mathscr{F}}{\delta\Omega}\frac{\delta^L\mathscr{F}}{\delta\Omega^{(Q)}} - A^{(s)}_\mu\frac{\delta^L\mathscr{F}}{\delta A^{(Qs)}_\mu} + \overline\lambda^{(s)}\frac{\delta^L\mathscr{F}}{\delta\overline\lambda^{(Qs)}} - \phi^{(s)}_i\frac{\delta^L\mathscr{F}}{\delta\phi^{(Qs)}_i} + \Omega^{(s)}\frac{\delta^L\mathscr{F}}{\delta\Omega^{(Qs)}} - b\frac{\delta^L\mathscr{F}}{\delta\bar c} + \bar\Omega\frac{\delta^L\mathscr{F}}{\delta\bar\mu}\\ -i(\bar\epsilon\gamma^\mu\epsilon)\Big(&-\partial_\mu A^{(Qs)}_\nu\frac{\delta^L\mathscr{F}}{\delta A^{(s)}_\nu} + \partial_\mu\overline\lambda^{(Qs)}\frac{\delta^L\mathscr{F}}{\delta\overline\lambda^{(s)}} - \partial_\mu\phi^{(Qs)}_i\frac{\delta^L\mathscr{F}}{\delta\phi^{(s)}_i} + \partial_\mu\Omega^{(Qs)}\frac{\delta^L\mathscr{F}}{\delta\Omega^{(s)}} - \partial_\mu\bar c\frac{\delta^L\mathscr{F}}{\delta b} + \partial_\mu\bar\mu\frac{\delta^L\mathscr{F}}{\delta\bar\Omega}\\ &+ A^{(Q)}_\nu\partial_\mu A^\nu + \overline\lambda^{(Q)}\partial_\mu\lambda + \phi^{(Q)}_i\partial_\mu\phi^i + \Omega^{(Q)}\partial_\mu\Omega + c^{(Q)}\partial_\mu c + \mu^{(Q)}\partial_\mu\mu\Big)\Big)\end{aligned} \tag{8}$$

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[3]Note that power counting forbids a gluino dependence for the argument of the $s\,Q$-exact term, and that Q is nilpotent on all the functionals that do not depend on the gluinos. α is the usual interpolating Landau–Feynman gauge parameter.

[4]M is the 32×32 matrix $M \equiv \frac{1}{2}(\bar\epsilon\gamma^\mu\epsilon)\gamma_\mu + \frac{1}{2}(\bar\epsilon\tau_i\epsilon)\tau^i - \epsilon\bar\epsilon$. It occurs because Q^2 is a pure derivative only modulo equations of motion. The dimension of A_μ, λ, ϕ^i, Ω, $\bar\Omega$, b, μ, $\bar\mu$, c and $\bar c$ are respectively 1, $\frac{3}{2}$, 1, 0, 2, 2, $\frac{1}{2}$, $\frac{3}{2}$, $\frac{1}{2}$ and $\frac{3}{2}$. Their ghost and shadow numbers are respectively $(0,0)$, $(0,0)$, $(0,0)$, $(1,0)$, $(-1,0)$, $(0,0)$, $(1,1)$, $(-1,-1)$, $(0,1)$ and $(0,-1)$.

[5]The linearized Slavnov–Taylor operator $\mathcal{S}_{(Q)|\Sigma}$ [9] verifies ${\mathcal{S}_{(Q)|\Sigma}}^2 = -i(\bar\epsilon\gamma^\mu\epsilon)\partial_\mu$, which solves in practice the fact that Q^2 is a pure derivative only modulo equations of motion.

2.2 Observables

The observables of the $\mathcal{N}=4$ super-Yang–Mills theory in the conformal phase are Green functions of local operators in the cohomology of the BRST linearized Slavnov–Taylor operator $\mathcal{S}_{(s)|\Sigma}$. From this definition, these Green functions are independent of the gauge parameters of the action, including ϵ. Classically, they are represented by gauge-invariant polynomials of the physical fields [9, 10]. We introduce classical sources u for all these operators. We must generalize the supersymmetry Slavnov–Taylor identity for the extended local action that depends on these sources. Since the supersymmetry algebra does not close off-shell, other sources v, coupled to unphysical $\mathcal{S}_{(s)|\Sigma}$-exact operators, must also be introduced. We define the following field and source combinations φ^*

$$\begin{aligned} A^*_\mu &\equiv A^{(Q)}_\mu - \partial_\mu \bar c - [A^{(Q_s)}_\mu - \partial_\mu \bar\mu, \Omega] \qquad & c^* &\equiv c^{(Q)} - [\mu^{(Q)}, \Omega] \\ \phi^*_i &\equiv \phi^{(Q)}_i - [\phi^{(Q_s)}_i, \Omega] & \lambda^* &\equiv \lambda^{(Q)} - [\lambda^{(Q_s)}, \Omega] \end{aligned} \tag{9}$$

They verify $\mathcal{S}_{(s)|\Sigma}\varphi^* = -[\Omega, \varphi^*]$. The collection of local operators coupled to the v's is made of all possible gauge-invariant (i.e. $\mathcal{S}_{(s)|\Sigma}$-invariant) polynomials in the physical fields and the φ^*'s. These operators have ghost number zero, and their shadow number is negative, in contrast with the physical gauge-invariant operators, which have shadow number zero.

The relevant action is thus $\Sigma[u,v] \equiv \Sigma + \Upsilon[u,v]$, with

$$\begin{aligned} \Upsilon[u,v] \equiv \int d^4x \Big(& u_{ij}\frac{1}{2}\mathrm{Tr}\ \phi^i\phi^j + u_i^\alpha \mathrm{Tr}\ \phi^i\lambda_\alpha + u_{ijk}\frac{1}{3}\mathrm{Tr}\ \phi^i\phi^j\phi^k \\ & + {}^{\mathrm{K}}u^\mu_{ij}\mathrm{Tr}\ \big(i\phi^{[i}D_\mu\phi^{j]} + \frac{1}{8}\overline{\lambda}\gamma_\mu\tau^{ij}\lambda\big) + {}^{\mathrm{K}}u^{\mu\nu}_i\mathrm{Tr}\ \big(F_{\mu\nu}\phi^i - \frac{1}{2}\overline{\lambda}\gamma_{\mu\nu}\tau^i\lambda\big) + {}^{\mathrm{K}}u^5_\mu\frac{1}{2}\mathrm{Tr}\ \overline{\lambda}\gamma_5\gamma^\mu\lambda \\ & + {}^{\mathrm{C}}u_{ijk}\mathrm{Tr}\ \big(\frac{1}{3}\phi^{[i}\phi^j\phi^{k]} + \frac{1}{8}\overline{\lambda}\tau^{ijk}\lambda\big) + {}^{\mathrm{C}}u^\mu_{ij}\mathrm{Tr}\ \big(i\phi^{[i}D_\mu\phi^{j]} - \frac{1}{4}\overline{\lambda}\gamma_\mu\tau^{ij}\lambda\big) \\ & + {}^{\mathrm{C}}u^{\mu\nu}_i\mathrm{Tr}\ \big(F_{\mu\nu}\phi^i + \frac{1}{4}\overline{\lambda}\gamma_{\mu\nu}\tau^i\lambda\big) + u^\alpha_{ij}\mathrm{Tr}\ \phi^i\phi^j\lambda_\alpha + iu^{\mu\,\alpha}_i\mathrm{Tr}\ D_\mu\phi^i\lambda_\alpha + u^{\mu\nu\,\alpha}\mathrm{Tr}\ F_{\mu\nu}\lambda_\alpha + \cdots \\ & + v^\alpha_i\mathrm{Tr}\ \phi^i\lambda^*_\alpha + v^{\alpha\beta}\mathrm{Tr}\ \lambda_\alpha\lambda^*_\beta + v^\mu_i\mathrm{Tr}\ \phi^iA^*_\mu + v_{ij}\mathrm{Tr}\ \phi^i\phi^{*\,j} + iv^{\mu\,\alpha}_i\mathrm{Tr}\ D_\mu\phi^i\lambda^*_\alpha \\ & + {}^{0}v^\alpha_i\mathrm{Tr}\ \lambda_\alpha\phi^{*\,i} + iv^{\mu\alpha\beta}\mathrm{Tr}\ D_\mu\lambda_\alpha\lambda^*_\beta + iv^\mu_{ij}\mathrm{Tr}\ D_\mu\phi^i\phi^{*\,j} + i\,{}^{-1}v^{\mu\alpha}_i\mathrm{Tr}\ D_\mu\lambda_\alpha\phi^{*\,i} + \cdots \Big) \end{aligned} \tag{10}$$

Here, the $\cdots$ stand for all other analogous operators.

The Slavnov–Taylor operator $\mathcal{S}_{(Q)}$ can be generalized into a new one, $\mathcal{S}^{\mathrm{ext}}_{(Q)}$, by addition of terms that are linear in the functional derivatives with respect to the sources u and v, in such a way that

$$\mathcal{S}^{\mathrm{ext}}_{(Q)}(\Sigma[u,v]) = \mathcal{S}_{(Q)}(\Sigma) + \mathcal{S}^{\mathrm{ext}}_{(Q)|\Sigma}\Upsilon + \int d^4x \mathrm{Tr}\ \left(\frac{\delta^R\Upsilon}{\delta A^\mu}\frac{\delta^L\Upsilon}{\delta A^*_\mu} + \frac{\delta^R\Upsilon}{\delta\lambda}\frac{\delta^L\Upsilon}{\delta\overline{\lambda}^*} + \frac{\delta^R\Upsilon}{\delta\phi^i}\frac{\delta^L\Upsilon}{\delta\phi^*_i}\right) = 0 \tag{11}$$

Indeed, if we were to compute $\mathcal{S}_{(Q)}(\Sigma[u,v])$ without taking into account the transformations of the sources u and v, the breaking of the Slavnov–Taylor identity would be a local functional linear in the set of gauge-invariant local polynomials in the physical fields, A^*_μ, c^*, ϕ^*_i and λ^*.

Eq. (11) defines the transformations $\mathcal{S}^{\rm ext}_{(Q)|\Sigma}$ of the sources u and v. Simplest examples for the transformation laws of the u's are for instance

$$\begin{aligned}
\mathcal{S}^{\rm ext}_{(Q)|\Sigma} u_{ij} &= -i[\gamma^\mu \tau_{\{i}\epsilon]_\alpha \partial_\mu u^\alpha_{j\}} + \partial_\mu\partial^\mu v_{\{ij\}} + 2u_{\{i|k} v_{j\}}{}^k + 2u^\alpha_{\{i} v_{j\}\alpha} - i\partial_\mu (u_{\{i|k} v^{\mu\ k}_{j\}} + u^\alpha_{\{i} v^\mu_{j\}\alpha}) \\
\mathcal{S}^{\rm ext}_{(Q)|\Sigma} u^\alpha_i &= [\overline{\epsilon}\tau^j]^\alpha \big(u_{ij} - i\partial_\mu ({}^{\rm K}u^\mu_{ij} + {}^{\rm C}u^\mu_{ij})\big) - 2i[\overline{\epsilon}\gamma_\mu]^\alpha \partial_\nu ({}^{\rm K}u^{\mu\nu}_i + {}^{\rm C}u^{\mu\nu}_i) + i[\gamma^\mu]_\beta{}^\alpha \partial_\mu v^\beta_i \\
&\qquad - u_{ij}\, {}^0 v^{j\alpha} - u^\alpha_j v_i{}^j + u^\beta_i v^\alpha{}_\beta + u^{\alpha\beta} v_{i\beta} + i\partial_\mu (u_{ij}\, {}^{-1}v^{j\mu\alpha} - u^\beta_i v^{\mu\alpha}_\beta)
\end{aligned} \tag{12}$$

These transformations are quite complicated in their most general expression. However, for many practical computations of non-supersymmetric local counterterms, we can consider them at $v=0$. We define $Qu \equiv \big(\mathcal{S}^{\rm ext}_{(Q)|\Sigma} u\big)_{|v=0}$. By using $\delta^{\mathcal{S}_{usy}} \Upsilon[u] + \Upsilon[Qu] = 0$ we can in fact conveniently compute Qu. Notice that Q is not nilpotent on the sources, but we have the result that $\Upsilon[Q^2 u]$ is a linear functional of the equation of motion of the fermion λ.

In [9, 11], we showed the absence of anomaly and the stability of the $\mathcal{N}=4$ action Σ under renormalization. Thus, the complete theory involving shadows and ghosts can be renormalized, in any given regularization scheme, so that supersymmetry and gauge invariance are preserved at any given finite order. It is a straightforward and precisely defined process to compute all observables, provided that a complete set of sources has been introduced. This lengthy process cannot be avoided because there exists no regulator that preserves both gauge invariance and supersymmetry. We must keep in mind that renormalization generally mixes physical observables with BRST-exact operators, and a careful analysis must be done [12].

3 Enforcement of supersymmetry

We now turn to the problem of determining non-invariant counterterms, which are necessary to ensure supersymmetry at the quantum level. We will use the notation of [7] for the 1PI correlation functions, an example of which is

$$\big\langle A^a_\mu(p) \lambda^b_\alpha(k) \phi^{(Q)\,c}_i \big\rangle \equiv \int d^4x d^4y\ e^{i(px+ky)}\ \frac{\delta^L\ \delta^L\ \delta^L\ \Gamma}{\delta\phi^{(Q)\,c\,i}(0)\delta\overline{\lambda}^{b\,\alpha}(y)\delta A^{a\,\mu}(x)} \tag{13}$$

All fields and sources are set equal to zero after the differentiation. Latin letters a, b, ... label the index of the gauge Lie algebra. In this section we focus on the case of observables that do not mix with non-gauge-invariant operators. The following subsection explains how computations are simplified in this case.

3.1 Loop cancellations

We can first eliminate by Gaussian integration the Faddeev–Popov ghosts Ω, $\bar{\Omega}$ against the shadows $\bar{\mu}$, μ for computing some observables, in our class of linear gauges.

The antighost Ward identities of [9] determine the following dependence of the 1PI generating functional Γ in the fields $\bar{\mu}$, $\bar{\Omega}$, $\bar{c}$ and b

$$\Gamma[\cdots,\bar{\mu},\bar{c},\bar{\Omega},b,A_\mu^{(s)},A_\mu^{(Q)},A_\mu^{(Q_s)}]=\Gamma[\cdots,0,0,0,0,A_\mu^{(s)}-\partial_\mu\bar{\Omega},A_\mu^{(Q)}+\partial_\mu\bar{c},A_\mu^{(Q_s)}+\partial_\mu\bar{\mu}]$$
$$-\int d^4x\mathrm{Tr}\,\Big(b\partial^\mu A_\mu+\frac{\alpha}{2}b^2-\frac{i\alpha}{2}(\bar{\epsilon}\gamma^\mu\epsilon)\bar{c}\partial_\mu\bar{c}\Big) \quad (14)$$

where the $\cdots$ stand for the dependence on all other fields and sources. Consider the generating functional of 1PI correlation functions of the subset of fields $\varphi_{\rm sub}$ made of the physical fields, the shadow c, $\bar{c}$ and the sources associated to the Q variations of these fields. The pair of Q-doublets Ω, μ and $\bar{\mu}$, $\bar{\Omega}$ only appear in the Feynman diagrams through propagators and interactions defined by the following part of the action

$$\int d^4x\mathrm{Tr}\,\big(\partial^\mu\bar{\Omega}D_\mu\Omega-\partial^\mu\bar{\mu}D_\mu\mu\big) \quad (15)$$

Feynman rules show that the fields Ω, $\bar{\Omega}$, μ and $\bar{\mu}$ exactly compensate in closed loops of opposite contributions at least at the regularized level. The following Ward identities imply that this property is maintained after renormalization

$$\big\langle\bar{\mu}^a(p)\mu^b\big\rangle+\big\langle\bar{\Omega}^a(p)\Omega^c\big\rangle\big\langle\Omega_c^{(Q)}(p)\mu^b\big\rangle=0$$
$$\big\langle\bar{\mu}^a(p)A_\mu^b(k)\mu^c\big\rangle+\big\langle\bar{\Omega}^a(p)A_\mu^b(k)\Omega^d\big\rangle\big\langle\Omega_d^{(Q)}(p+k)\mu^c\big\rangle=0 \quad (16)$$

$\Omega^{(Q)}$ is the source of the operator $\mu+[\Omega,c]$ and the term linear in μ of the insertion of $[\Omega,c]$ in Γ must be zero. It follows that $\big\langle\Omega_c^{(Q)}(p)\mu^b\big\rangle=\delta_c^b$, at any given finite order of perturbation theory. The only superficially divergent 1PI Green functions depending on Ω, $\bar{\Omega}$, μ and $\bar{\mu}$ that must be considered are $\big\langle\bar{\mu}^a(p)\mu^b\big\rangle=-\big\langle\bar{\Omega}^a(p)\Omega^b\big\rangle$ and $\big\langle\bar{\mu}^a(p)A_\mu^b(k)\mu^c\big\rangle=-\big\langle\bar{\Omega}^a(p)A_\mu^b(k)\Omega^c\big\rangle$. We can thus integrate out these fields in all correlation functions of the fields $\varphi_{\rm sub}$.

After this elimination, the supersymmetry Slavnov–Taylor identity is sufficient to constrain the 1PI Green functions of the fields $\varphi_{\rm sub}$ to the same values as they would have in the complete procedure without the ab-initio elimination of Ω, $\bar{\Omega}$, μ and $\bar{\mu}$. In fact, the most general local functional, which satisfies the supersymmetry Slavnov–Taylor identity and only depends on the fields $\varphi_{\rm sub}$, is the restriction of the most general local functional that satisfies all the Ward identities of the theory when all the other fields and sources are set equal to zero. This justifies the heuristic argument that the Ward identity of supersymmetry implies that of gauge invariance because supersymmetry transformations close modulo gauge transformations. Thus, to compute correlation functions of fields $\varphi_{\rm sub}$, we can use the simplified action

$$\Sigma^{\rm sub} \equiv \frac{1}{g^2} S + Q \int d^4x \text{Tr} \left(\bar{c} \partial^\mu A_\mu + \frac{\alpha}{2} \bar{c} b \right) \\ + \int d^4x \text{Tr} \left(A^{(Q)}_\mu Q A^\mu - \overline{\lambda}^{(Q)} Q \lambda + \phi^{(Q)}_i Q \phi^i - c^{(Q)} Q c + \frac{g^2}{2} \overline{\lambda}^{(Q)} M \lambda^{(Q)} \right) \tag{17}$$

The ambiguities of the quantum theory are fixed by the antighost Ward identities for $\bar{c}$ and b, and the following simplified supersymmetry Slavnov–Taylor identity

$$\mathcal{S}_{(Q)}(\mathscr{F}) \equiv \int d^4x \text{Tr} \left(\frac{\delta^R \mathscr{F}}{\delta A^\mu} \frac{\delta^L \mathscr{F}}{\delta A^{(Q)}_\mu} + \frac{\delta^R \mathscr{F}}{\delta \lambda} \frac{\delta^L \mathscr{F}}{\delta \overline{\lambda}^{(Q)}} + \frac{\delta^R \mathscr{F}}{\delta \phi^i} \frac{\delta^L \mathscr{F}}{\delta \phi^{(Q)}_i} + \frac{\delta^R \mathscr{F}}{\delta c} \frac{\delta^L \mathscr{F}}{\delta c^{(Q)}} \right. \\ \left. - i (\overline{\epsilon} \gamma^\mu \epsilon) \Big(A^{(Q)}_\nu \partial_\mu A^\nu + \overline{\lambda}^{(Q)} \partial_\mu \lambda + \phi^{(Q)}_i \partial_\mu \phi^i + c^{(Q)} \partial_\mu c \Big) - b \frac{\delta^L \mathscr{F}}{\delta \bar{c}} + i (\overline{\epsilon} \gamma^\mu \epsilon) \partial_\mu \bar{c} \frac{\delta^L \mathscr{F}}{\delta b} \right) \tag{18}$$

This identity is analogous to that in [5, 6, 8]. However, we now understand that c is not the Faddeev–Popov ghost, and that observables must be defined in the enlarged theory.

This simplified process with less fields can be applied also for computing 1PI correlation functions with insertions of certain physical composite operators (we call them "simple" operators), as long as these operators do not mix through renormalization with BRST-exact operators (which would imply computing insertions of operators depending on other fields than the $\varphi_{\rm sub}$). At the tree level, these "simple" operators are all the gauge-invariant polynomials in the physical fields that are in representations of $Spin(3,1) \times SU(4)$ in which there exist no BRST-exact operators of the same canonical dimensions that depend on the antighost $\bar{\mu}$, $\bar{\Omega}$ and $\bar{c}$ only through their derivatives. Examples of "simple" operators are the local operators of canonical dimension $[\mathcal{O}] < 4$ and the BPS primary operators.

3.2 Renormalization of the action

We assume that the "restricted" theory has been renormalized at a given order of perturbation theory, say n, by using the best available regularization, namely dimensional reduction, and renormalization conditions such that the supersymmetry Slavnov–Taylor identity and the so-called antighost Ward identities are satisfied. Within this scheme, finite gauge-invariant, but not supersymmetric, counterterms must occur after a certain order of perturbation theory. At a given order n, the action is thus of the following form

$$\begin{aligned}
\Sigma^\flat = \frac{1}{g^2}\int d^4x \mathrm{Tr}\,\Big(&-\frac{1}{4}F^\flat_{\mu\nu}F^{\flat\,\mu\nu} - D^\flat_\mu\phi^{\flat\,i}D^{\flat\,\mu}\phi^\flat_i + \frac{i}{2}\big(\overline{\lambda}^\flat \not{D}^\flat\lambda^\flat\big) \\
&- \frac{g_1}{2}\big(\overline{\lambda}^\flat[\phi^\flat,\lambda^\flat]\big) - \frac{g_2}{4}[\phi^{\flat\,i},\phi^{\flat\,j}][\phi^\flat_i,\phi^\flat_j] + h_1\phi^{\flat\,\{i}\phi^{\flat\,j\}}\phi^\flat_{\{i}\phi^\flat_{j\}} + h_2\phi^{\flat\,i}\phi^\flat_i\phi^{\flat\,j}\phi^\flat_j\Big) \\
&\qquad + \int d^4x\Big(g_3\mathrm{Tr}\ \phi^{\flat\,\{i}\phi^{\flat\,j\}}\mathrm{Tr}\ \phi^\flat_{\{i}\phi^\flat_{j\}} + h_3\mathrm{Tr}\ \phi^{\flat\,i}\phi^\flat_i\mathrm{Tr}\ \phi^{\flat\,j}\phi^\flat_j\Big) \\
+ \int d^4x \mathrm{Tr}\,\Big(&-b^\flat\partial^\mu A^\flat_\mu - \frac{\alpha^\flat}{2}{b^\flat}^2 + \overline{c}^\flat\partial^\mu\big(D^\flat_\mu c^\flat + i\mathrm{y}_1(\overline{\epsilon}\gamma_\mu\lambda^\flat)\big) + \mathrm{y}_2\frac{i\alpha^\flat}{2}(\overline{\epsilon}\gamma^\mu\epsilon)\overline{c}^\flat\partial_\mu\overline{c}^\flat\Big) \\
+ \int d^4x \mathrm{Tr}\,\Big(&A^{(Q)\flat}_\mu\big(i\mathrm{x}_1(\overline{\epsilon}\gamma^\mu\lambda^\flat) + D^{\flat\,\mu}c^\flat\big) - \phi^{(Q)\,\flat}_i\big(\mathrm{x}_2(\overline{\epsilon}\tau^i\lambda^\flat) + [c^\flat,\phi^{\flat\,i}]\big) \\
&+ \overline{\lambda}^{(Q)\,\flat}\big(-\big[\mathrm{x}_3\not{F}^\flat + i\mathrm{x}_4\not{D}^\flat\phi^\flat + \frac{\mathrm{x}_5}{2}[\phi^\flat,\phi^\flat] + h_4\phi^{\flat\,i}\phi^\flat_i\big]\epsilon + [c^\flat,\lambda^\flat]\big) \\
&\qquad + c^{(Q)\,\flat}\big(-\mathrm{x}_6(\overline{\epsilon}\phi^\flat\epsilon) + i\mathrm{x}_7(\overline{\epsilon}\not{A}^\flat\epsilon) + {c^\flat}^2\big) + \frac{g^2}{2}\overline{\lambda}^{(Q)\,\flat}N\lambda^{(Q)\,\flat}\Big)
\end{aligned} \tag{19}$$

The conformal property of $\mathcal{N}=4$ implies that the coupling constant is not renormalized. In this expression, the index $\flat$ on top of a field φ indicates its multiplicative renormalization by an infinite factor $\sqrt{Z_\varphi}$, which is a Taylor series of order n in the coupling constant g^2. The sources $\varphi^{(Q)}$ are renormalized by the inverse factor $1/\sqrt{Z_\varphi}$ as a result of the BRST Slavnov–Taylor identities. The parameters g_I, h_I, x_I, y_I and the 32×32 symmetric matrix N (quadratic in ϵ)[6] are finite power series in g^2 of order n, which have been fined-tuned to enforce supersymmetry. In the simplest case of the $SU(2)$ gauge group, the parameters h_I are redundant and can be set to zero. This action permits us to perturbatively compute the renormalized 1PI generating functional Γ_n of the φ_{sub} at order n, such that the Slavnov–Taylor identity of supersymmetry is verified at this order. To obtain the action (19) at the following order $n+1$, we then use the minimal subtraction scheme with dimensional reduction, which defines the infinite factors Z_φ at order $n+1$. They yield as an intermediary result the "minimally" renormalized 1PI generating functional $\Gamma^{\min}_{n+1} = \sum_{p=0}^n \Gamma_{(p)} + \Gamma^{\min}_{(n+1)}$. The supersymmetry Slavnov–Taylor

[6] N can be parametrized by five parameters as follows

$$N \equiv \mathrm{a}_1(\overline{\epsilon}\gamma^\mu\epsilon)\gamma_\mu + \mathrm{a}_2(\overline{\epsilon}\tau^i\epsilon)\tau_i + 6\mathrm{a}_3(\overline{\epsilon}\gamma^{\mu\nu}\tau^{ijk}\epsilon)\gamma_{\mu\nu}\tau_{ijk} + 2\mathrm{a}_4(\overline{\epsilon}\gamma_5\gamma^\mu\tau^{ij}\epsilon)\gamma_5\gamma_\mu\tau_{ij} + \mathrm{a}_5(\overline{\epsilon}\gamma_5\tau^i\epsilon)\gamma_5\tau_i$$

identity is possibly broken at $n+1$ order, as follows[7]

$$\mathcal{S}_{(Q)}(\Gamma^{\min}_{n+1}) = \frac{1}{2}\sum_{p=1}^{n}\left(\Gamma_{(p)},\Gamma_{(n+1-p)}\right) + \mathcal{S}_{(Q)|\Sigma}\Gamma^{\min}_{(n+1)} + \mathcal{O}(g^{2n+4}) \tag{20}$$

Any given term in the right-hand side of Eq. (20) may be non-local, but the sum of these terms is a local functional of fields and sources, as is warranted by the quantum action principle. There is no supersymmetry anomaly [6, 9] and the consistency relation $\mathcal{S}_{(Q)|\Gamma}\mathcal{S}_{(Q)}(\Gamma)=0$ implies the existence of the local functional $\Sigma^{\rm corr}_{(n+1)}$ such that $\mathcal{S}_{(Q)}(\Gamma^{\min}_{n+1}+\Sigma^{\rm corr}_{(n+1)}) = \mathcal{O}(g^{2n+4})$. Thus the component of order $n+1$ of the parameters g_I, h_I, x_I, y_I and the matrix N can be modified in such a way that the resulting 1PI generating functional Γ_{n+1} satisfies the supersymmetry Slavnov–Taylor identity at order $n+1$.

The fine-tuning at order $n+1$ will be achieved if a large enough number of relations between 1PI Green functions are satisfied. They are obtained by suitable differentiations of the supersymmetry Slavnov–Taylor identity. The number of ambiguities removed by the Slavnov–Taylor identity is finite and corresponds to that of parameters of the action. Thus the relations between the 1PI Green functions only have to be implemented on their renormalization conditions. These relations must be expanded on Lorentz and gauge group invariant tensors.

The antighost Ward identities fix the ambiguities on the Green functions that contain the antishadow $\bar{c}$ and the b field. The identities

$$\begin{aligned}\left\langle \bar{c}^a(p)\lambda^b_\alpha\right\rangle &= -ip^\mu\left\langle A^{(Q)\,a}_\mu(p)\lambda^b_\alpha\right\rangle\\ \left\langle \bar{c}^a(p)\bar{c}^b\right\rangle &= \alpha(\bar{\epsilon}\not{p}\epsilon)\delta^{ab} - ip^\mu\left\langle A^{(Q)\,a}_\mu(p)\bar{c}^b\right\rangle + ip^\mu\left\langle A^{(Q)\,b}_\mu(-p)\bar{c}^a\right\rangle\end{aligned} \tag{21}$$

permit to compute the value of y_1 in function of x_1, and y_2 at the $n+1$ order.

We first use the components of the Slavnov–Taylor identity that expresses the closure of the supersymmetry algebra at the quantum level

$$\begin{aligned}&\left\langle A^{(Q)\,a}_\mu(p)\overline{\lambda}^{\alpha\, c}\right\rangle\left\langle\lambda^{(Q)}_{\alpha\, c}(p)A^b_\nu\right\rangle + \left\langle A^{(Q)\,a}_\mu(p)c^c\right\rangle\left\langle c^{(Q)}_c(p)A^b_\nu\right\rangle + (\bar{\epsilon}\not{p}\epsilon)\delta^{ab}\eta_{\mu\nu}\\ &\qquad + \left\langle A^b_\nu(-p)A^c_\sigma\right\rangle\left\langle A^{(Q)\,\sigma}_c(-p)A^{(Q)\,a}_\mu\right\rangle = 0\end{aligned} \tag{22}$$

$$\left\langle c^{(Q)\,a}(p)A^c_\mu\right\rangle\left\langle A^{(Q)\,\mu}_c(p)\lambda^b_\alpha\right\rangle + \left\langle c^{(Q)\,a}(p)\phi^c_i\right\rangle\left\langle \phi^{(Q)\,i}_c(p)\lambda^b_\alpha\right\rangle + \left\langle\lambda^b_\alpha(-p)\overline{\lambda}^{\beta\, c}\right\rangle\left\langle\lambda^{(Q)}_{\beta\, c}(-p)c^{(Q)\,a}\right\rangle = 0$$

[7] $(\mathscr{F},\mathcal{G})$ is the antibracket

$$\int d^4x\mathrm{Tr}\left(\frac{\delta^R\mathscr{F}}{\delta A^\mu}\frac{\delta^L\mathcal{G}}{\delta A^{(Q)}_\mu} + \frac{\delta^R\mathscr{F}}{\delta\lambda}\frac{\delta^L\mathcal{G}}{\delta\overline{\lambda}^{(Q)}} + \frac{\delta^R\mathscr{F}}{\delta\phi^i}\frac{\delta^L\mathcal{G}}{\delta\phi^{(Q)}_i} + \frac{\delta^R\mathscr{F}}{\delta c}\frac{\delta^L\mathcal{G}}{\delta c^{(Q)}} - \frac{\delta^R\mathscr{F}}{\delta A^{(Q)}_\mu}\frac{\delta^L\mathcal{G}}{\delta A^\mu} + \frac{\delta^R\mathscr{F}}{\delta\lambda^{(Q)}}\frac{\delta^L\mathcal{G}}{\delta\overline{\lambda}} - \frac{\delta^R\mathscr{F}}{\delta\phi^{(Q)}_i}\frac{\delta^L\mathcal{G}}{\delta\phi^i} - \frac{\delta^R\mathscr{F}}{\delta c^{(Q)}}\frac{\delta^L\mathcal{G}}{\delta c}\right)$$

$$\left\langle A_\mu^{(Q)\,a}(p)\overline{\lambda}^{\alpha\,c}\right\rangle\left\langle\lambda_{\alpha\,c}^{(Q)}(p)\phi_i^b\right\rangle+\left\langle A_\mu^{(Q)\,a}(p)c^c\right\rangle\left\langle c_c^{(Q)}(p)\phi_i^b\right\rangle+\left\langle\phi_i^b(-p)\phi_j^c\right\rangle\left\langle\phi_c^{(Q)\,j}(-p)A_\mu^{(Q)\,a}\right\rangle=0$$

$$\begin{aligned}&\left\langle\phi_k^{(Q)\,c}(p)\overline{\lambda}^{\alpha\,d}\right\rangle\left\langle\lambda_{\alpha\,d}^{(Q)}(p)\phi_i^a(k)\phi_j^b\right\rangle+\left\langle\phi_k^{(Q)\,c}(p)\phi_i^a(k)c^d\right\rangle\left\langle c_d^{(Q)}(p+k)\phi_j^b\right\rangle\\&+\left\langle\phi_k^{(Q)\,c}(p)\phi_j^b(-p-k)c^d\right\rangle\left\langle c_d^{(Q)}(-k)\phi_i^a\right\rangle+\left\langle\phi_i^a(k)\phi_j^b(-p-k)A_\mu^d\right\rangle\left\langle A_d^{(Q)\,\mu}(-k)\phi_k^{(Q)\,c}\right\rangle=0\end{aligned}$$

These identities imply that the quantities x_I are functions of only two independent parameters. In turn, both parameters are determined from the following Slavnov–Taylor identities, which express the supersymmetry covariance of physical Green functions

$$\left\langle A_\mu^a(p)A_\nu^c\right\rangle\left\langle A_c^{(Q)\,\nu}(p)\lambda_\alpha^b\right\rangle+\left\langle\lambda_\alpha^b(-p)\overline{\lambda}^{\beta\,c}\right\rangle\left\langle\lambda_{\beta\,c}^{(Q)}(-p)A_\mu^a\right\rangle=0$$

$$\begin{aligned}&\left\langle\phi_i^a(p)\phi_j^b(k)A_\mu^d\right\rangle\left\langle A_d^{(Q)\,\mu}(p+k)\lambda_\alpha^c\right\rangle+\left\langle\lambda_\alpha^c(-p-k)\overline{\lambda}^{\beta\,d}\right\rangle\left\langle\lambda_{\beta\,d}^{(Q)}(-p-k)\phi_i^a(p)\phi_j^b\right\rangle\\&+\left\langle\lambda_\alpha^c(-p-k)\phi_i^a(p)\overline{\lambda}^{\beta\,d}\right\rangle\left\langle\lambda_{\beta\,d}^{(Q)}(-k)\phi_j^b\right\rangle+\left\langle\lambda_\alpha^c(-p-k)\phi_j^b(k)\overline{\lambda}^{\beta\,d}\right\rangle\left\langle\lambda_{\beta\,d}^{(Q)}(-p)\phi_i^a\right\rangle=0\end{aligned}\quad(23)$$

$$\begin{aligned}&\left\langle\phi_{i_1}^{a_1}(p_1)\phi_{i_2}^{a_2}(p_2)\phi_{i_3}^{a_3}(p_3)\phi_j^c\right\rangle\left\langle\phi_c^{(Q)\,j}(p_1+p_2+p_3)\lambda_\alpha^b\right\rangle\\&\quad+\sum_{r\in\mathbb{Z}_3}\left\langle\lambda_\alpha^b(-p_1-p_2-p_3)\phi_{i_{1+r}}^{a_{1+r}}(p_{1+r})\phi_{i_{2+r}}^{a_{2+r}}(p_{2+r})\overline{\lambda}^{\beta\,c}\right\rangle\left\langle\lambda_{\beta\,c}^{(Q)}(-p_{3+r})\phi_{i_{3+r}}^{a_{3+r}}\right\rangle\\&\quad+\sum_{r\in\mathbb{Z}_3}\left\langle\lambda_\alpha^b(-p_1-p_2-p_3)\phi_{i_{3+r}}^{a_{3+r}}(p_{3+r})\overline{\lambda}^{\beta\,c}\right\rangle\left\langle\lambda_{\beta\,c}^{(Q)}(-p_{1+r}-p_{2+r})\phi_{i_{1+r}}^{a_{1+r}}(p_{1+r})\phi_{i_{2+r}}^{a_{2+r}}\right\rangle\\&\quad+\left\langle\lambda_\alpha^b(-p_1-p_2-p_3)\overline{\lambda}^{\beta\,c}\right\rangle\left\langle\lambda_{\beta\,c}^{(Q)}(-p_1-p_2-p_3)\phi_{i_1}^{a_1}(p_1)\phi_{i_2}^{a_2}(p_2)\phi_{i_3}^{a_3}\right\rangle=0\end{aligned}$$

It remains to determine the matrix N, which is related to the terms quadratic in the sources. This can be done using the identity

$$\begin{aligned}&\left\langle\lambda_\beta^b(-p)\overline{\lambda}^{\gamma\,c}\right\rangle\left\langle\lambda_{\gamma\,c}^{(Q)}(-p)\overline{\lambda}^{(Q)\,\alpha\,a}\right\rangle+\left\langle\overline{\lambda}^{(Q)\,\alpha\,a}(p)A_\mu^c\right\rangle\left\langle A_c^{(Q)\,\mu}(p)\lambda_\beta^b\right\rangle\\&\qquad+\left\langle\overline{\lambda}^{(Q)\,\alpha\,a}(p)\phi_i^c\right\rangle\left\langle\phi_c^{(Q)\,i}(p)\lambda_\beta^b\right\rangle+(\overline{\epsilon}\not{p}\epsilon)\delta^{ab}\delta_\beta^\alpha=0\end{aligned}\quad(24)$$

3.3 Renormalization of local observables

We must also renormalize the part of the action that is linear in the sources u of the local observables. Consider a set of local operators that mix together by renormalisation. Suppose that each one of these operators can be considered as the element of an irreducible supersymmetry multiplet. Then, all the other components of the supersymmetry multiplets will mix by renormalisation with the same matrix of anomalous dimensions. As for the ordinary Green functions, non-supersymmetric counterterms must be perturbatively computed for enforcing the Ward identities. The method of the preceding section can be generalized. We decompose each source into irreducible representations of $Spin(3,1) \times SU(4)$ and write the most general gauge-invariant functional linear in the sources.

$$
\begin{aligned}
\Upsilon^\flat[u,v] = \int d^4x \Big(& Z^{\mathrm{K}}\, u^i{}_i \frac{1}{2}\mathrm{Tr}\ \phi^{\flat\, j}\phi^\flat_j + Z^{\mathrm{C}}\, u_{ij}\frac{1}{2}\mathrm{Tr}\ \big(\phi^{\flat\, i}\phi^{\flat\, j} - \frac{1}{6}\delta^{ij}\phi^{\flat\, k}\phi^\flat_k\big) + Z_1^{\mathrm{K}}\, \overline{u}_i\tau^i \mathrm{Tr}\ \phi^\flat\lambda^\flat \\
& + Z_1^{\mathrm{C}}\, u_i^\alpha \mathrm{Tr}\ \big(\phi^{\flat\, i}\lambda^\flat - \frac{1}{6}\tau^i\phi^\flat\lambda^\flat\big) + Z_2^{\mathrm{K}}\, u_{[ijk]}\mathrm{Tr}\ \phi^{\flat\, i}[\phi^{\flat\, j},\phi^{\flat\, k}] + Z_2^{\mathrm{KC}}\, u_{[ijk]}\mathrm{Tr}\ \big(\frac{1}{3}\phi^{\flat\, i}\phi^{\flat\, j}\phi^{\flat k} + \frac{1}{8}\overline{\lambda}^\flat\tau^{ijk}\lambda^\flat\big) \\
& + Z_2^{\mathrm{CK}}\ {}^{\mathrm{C}}u_{ijk}\mathrm{Tr}\ \phi^{\flat\, i}[\phi^{\flat\, j},\phi^{\flat\, k}] + Z_2^{\mathrm{C}}\ {}^{\mathrm{C}}u_{ijk}\mathrm{Tr}\ \big(\frac{1}{3}\phi^{\flat\, i}\phi^{\flat\, j}\phi^{\flat\, k} + \frac{1}{8}\overline{\lambda}^\flat\tau^{ijk}\lambda^\flat\big) + \cdots \Big)
\end{aligned} \tag{25}
$$

There is an ambiguity corresponding to each one of the renormalization factors $Z_I^\bullet$, to be fixed by the supersymmetry Slavnov–Taylor identity. At a given order, we first perturbatively compute the infinite part of the renormalization factors. Then the finite part of the renormalization factors must be adjusted, as for ordinary Green functions.

Consider as the simplest cases the Konishi operator $\mathcal{O}^{\mathrm{K}} \equiv \frac{1}{2}\mathrm{Tr}\ \phi^i\phi_i$ and the $\frac{1}{2}$ BPS operator $\mathcal{O}^{ij}_{\mathrm{C}} \equiv \frac{1}{2}\mathrm{Tr}\ (\phi^i\phi^j - 1/6\,\delta^{ij}\phi^k\phi_k)$. The renormalization factors of the first two components of the associated supermultiplets are related because of the Ward identity

$$
\begin{aligned}
[\overline{\epsilon}\not{p}\tau^j]^\alpha \big\langle u_{ij}(p)\phi^a_k(k)\phi^b_l\big\rangle + \big\langle u^\alpha_i(p)\phi^a_k(k)\overline{\lambda}^{\beta\, c}\big\rangle\big\langle\lambda^{(Q)}_{\beta\, c}(p+k)\phi^b_l\big\rangle & \\
+ \big\langle u^\alpha_i(p)\phi^b_l(-p-k)\overline{\lambda}^{\beta\, c}\big\rangle\big\langle\lambda^{(Q)}_{\beta\, c}(-k)\phi^a_k\big\rangle &= 0
\end{aligned} \tag{26}
$$

A less simple example, for which there could be non-supersymmetric couterterms with a mixing-matrix, is for the cubic operator Tr $\phi^i[\phi^j,\phi^k]$ of the Konishi multiplet. The global symmetries and power counting allow this operator to mix with Tr $\big(\frac{1}{3}\phi^{[i}\phi^j\phi^{k]} + \frac{1}{8}\overline{\lambda}\tau^{ijk}\lambda\big)$ belonging to the $\frac{1}{2}$ BPS multiplet associated to $\mathcal{O}^{ij}_{\mathrm{C}}$. The identity

$$
\begin{aligned}
3[\overline{\epsilon}\tau^{klm}\tau_j]^\alpha & \big\langle u_{klm}(-p_1-p_2-p_3)\phi^{a_1}_{i_1}(p_1)\phi^{a_2}_{i_2}(p_2)\phi^{a_3}_{i_3}\big\rangle \\
& + 3[\overline{\epsilon}\tau_j\tau^{klm}]^\alpha\big\langle\ {}^{\mathrm{C}}u_{klm}(-p_1-p_2-p_3)\phi^{a_1}_{i_1}(p_1)\phi^{a_2}_{i_2}(p_2)\phi^{a_3}_{i_3}\big\rangle \\
+ \sum_{r\in\mathbb{Z}_3} & \big\langle u^\alpha_j(-p_1-p_2-p_3)\phi^{a_{3+r}}_{i_{3+r}}(p_{3+r})\overline{\lambda}^{\beta\, b}\big\rangle\big\langle\lambda^{(Q)}_{\beta\, b}(-p_{1+r}-p_{2+r})\phi^{a_{1+r}}_{i_{1+r}}(p_{1+r})\phi^{a_{2+r}}_{i_{2+r}}\big\rangle = 0
\end{aligned}
$$

and the component in $[\tau^{mnp}]_{\alpha\beta}$ of the following one

$$
\begin{aligned}
3[\overline{\epsilon}\tau^{jkl}\tau_i]^\gamma & \big\langle u_{jkl}(p)\lambda^a_\alpha(k)\lambda^b_\beta\big\rangle + 3[\overline{\epsilon}\tau_i\tau^{jkl}]^\gamma\big\langle\ {}^{\mathrm{C}}u_{jkl}(p)\lambda^a_\alpha(k)\lambda^b_\beta\big\rangle \\
& + \big\langle u^\gamma_i(p)\lambda^a_\alpha(k)\phi^c_j\big\rangle\big\langle\phi^{(Q)\, j}_c(p+k)\lambda^b_\beta\big\rangle - \big\langle u^\gamma_i(p)\lambda^b_\beta(-p-k)\phi^c_j\big\rangle\big\langle\phi^{(Q)\, j}_c(-k)\lambda^a_\alpha\big\rangle = 0
\end{aligned} \tag{27}
$$

permit us to determine perturbatively the renormalization factors of these operators in function of those of $\mathcal{O}^{\mathrm{K}}$ and $\mathcal{O}^{ij}_{\mathrm{C}}$.

In fact, the renormalization factors of all the other components of the supermultiplet containing $\mathcal{O}^{\rm K}$ and $\mathcal{O}^{ij}_{\rm C}$ can be perturbatively computed as a function of those of the operators $\mathcal{O}^{\rm K}$ and $\mathcal{O}^{ij}_{\rm C}$.

3.4 Contact terms

After computing the renormalization of one insertion of "simple" physical operators in all Green functions of fields $\varphi_{\rm sub}$, we may want to compute their multicorrelators. The renormalization of these correlation functions possibly involves the addition of contact terms. Such counterterms cannot be generated in the minimal scheme prescription. However, dimensional reduction breaks supersymmetry, and we expect that finite contact-counterterms must be added to the action, for restoring supersymmetry. To compute these possible counterterms, we write the more general $Spin(3,1) \times SU(4)$-invariant action that depends only on the sources u, in a polynomial way

$$\Xi[u] = \frac{1}{2}\int d^4x \Big({\rm z}_1 u^i{}_i u^j{}_j + {\rm z}_2 u^{ij} u_{ij} - i{\rm z}_3 \overline{u}_i \partial\!\!\!/ u^i - i{\rm z}_4 \overline{u}_i \tau^{ij} \partial\!\!\!/ u_j + {\rm z}_5 u^{[ijk]} \partial^\mu \partial_\mu u_{[ijk]} \\ + {\rm z}_6 u^{\{ijk\}} \partial^\mu \partial_\mu u_{\{ijk\}} + {\rm z}_7 u_j{}^{ji} \partial^\mu \partial_\mu u^k{}_{ki} + {\rm z}_8 u^\mu_{ij} \partial^\nu \partial_\nu u^{ij}_\mu + \cdots \Big) \qquad (28)$$

The values of the renormalization factors ${\rm z}_{\rm I}$ can then be computed, by imposing the supersymmetry Slavnov–Taylor identity, order by order in perturbation theory. As before, it is sufficient to enforce some identities between relevant correlation functions. The simplest identity

$$[p\!\!\!/ \tau_l \epsilon]_\alpha \big\langle u^{kl}(p) u^{ij} \big\rangle + [\overline{\epsilon} \tau^{\{i}]^\beta \big\langle u^{j\}}_\beta(p) u^k_\alpha \big\rangle = 0 \qquad (29)$$

constrains ${\rm z}_3$ and ${\rm z}_4$ as functions of ${\rm z}_1$ and ${\rm z}_2$, and so on. In practice, we have to define renormalization conditions for each one of the classes of superficially divergent correlation functions that are not related by the supersymmetry Slavnov–Taylor identity. Within a given class, the renormalization conditions of all correlation functions are related by supersymmetry Slavnov–Taylor identities. The non-invariant contact-counterterms can then be perturbatively computed by perturbatively enforcing these renormalization conditions.

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